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Torsion-Free Relatively Homogeneous Permutation Groups

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1. INTRODUCTION

In [1, Theorem 4], we constructed a faithful, sharp, relatively homogeneous, infinite group that is not locally finite. In answer to a question of P. M. Neumann's, we now strengthen the construction as follows:

THEOREM 1. *There exists a faithful, countably infinite permutation group* (G, Σ) *of countably infinite degree such that*

(i) (G, Σ) *is torsion-free,*

(ii) (G, Σ) *is sharp and relatively homogenous, and*

(iii) *the orbits of every finitely generated subgroup of* G *are finite sets.*

For more background on the topic, we suggest the reader consults [1]. However, in order to make this paper a bit self-contained, we restate the following definitions which originate in work by Kenneth Hickin:

DEFINITIONS. Let Ω be an infinite set and let G be a group such that $G \subseteq \mathrm{Sym}(\Omega)$, the set of all permutations of Ω.

(i) (G, Ω) is said to be *sharp* if for every g in $G \setminus \{1\}$, the set $\mathrm{fix}(g)$ of all elements fixed by g is empty or finite.

(ii) (G, Ω) is *relatively homogeneous* if for every finitely generated subgroup $B \subseteq G$, and every B-isomorphism f among finitely many orbits of B, there exists x in the centralizer $C_G(B)$ which extends f.

In Section 2 below we prove some lemmas which we use to prove Theorem 1 in Section 3.

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2. Preliminaries

(a) Definition of Group H. Let (x_i) $i \in \mathbb{N}$, be a sequence of symbols. For every $k \in \mathbb{N}$, let $F_k := F(x_1, x_2, \ldots, x_k)$ denote the free group generated by $x_1, x_2, \ldots, x_k$ and let $F_\infty := F(x_i \mid i \in \mathbb{N})$ denote the free group generated by the whole sequence (x_i) $i \in \mathbb{N}$. Also let (w_i) $i \in \mathbb{N} \setminus \{1\}$ be a sequence of all words in F_∞, where $w_i \in F_{i-1}$ for $i = 2, 3, 4, \ldots$. Define $(B(1, j))$ $j \in \mathbb{N}$ as the sequence of all finitely generated subgroups of F_1 and for each $i > 2$, define $(B(i, j))$ $j \in \mathbb{N}$ as the sequence of all finitely generated subgroups of F_i which are not contained in F_{i-1}. Let $\underset{\sim}{\theta}\colon \mathbb{N} \setminus \{1\} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ be a *bijective* map where we use the notations

$$\underset{\sim}{\theta}(i) := (\theta(i)_1, \theta(i)_2, \theta(i)_3)$$

and where $\theta(i)_1 < i$ for each $i \geqslant 2$. So, $\underset{\sim}{\theta}$ is an *enumeration* of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

Note that since $\theta(i)_1 < i$ for each $i \geqslant 2$, then we have $B(\theta(i)_1, \theta(i)_2) \subseteq F_{i-1}$. Define $K_1 := \{1\}$. For each $n > 1$, define K_n to be the normal closure in F_n of the subgroup generated by K_{n-1} and the subset $\{x_i b_i x_i^{-1} b_i^{-1} \mid b_i \in B(\theta(i)_1, \theta(i)_2) \quad i = 2, 3, \ldots, n\}$. This definition makes $K_1 < K_2 < K_3 < \cdots$.

Now define

$$H_1 := F_1$$

$$A_2 := B(\theta(2)_1, \theta(2)_2)$$

$$C(\theta(2)_1, \theta(2)_2) := A_2$$

$$H_2 := \langle H_1, x_2 \mid x_2 a x_2^{-1} a^{-1} = 1,\ \forall a \in A_2 \rangle.$$

Then H_2 is isomorphic to F_2/K_2 under a natural isomorphism. Let A_3 be the subgroup of H_2 that corresponds to $[B(\theta(3)_1, \theta(3)_3)\, K_2]/K_2$ under this isomorphism. For later reference, put $C(\theta(3)_1, \theta(3)_2) := A_3$. Define

$$H_3 := \langle H_2, x_3 \mid x_3 a x_3^{-1} a^{-1} = 1,\ \forall a \in A_3 \rangle.$$

We now define H_n inductively for $n = 4, 5, 6, \ldots$

Assume H_{n-1} is already defined, where $n > 3$. Assume also that A_n has been defined as the subgroup of H_{n-1} that corresponds to $(B(\theta(n)_1, \theta(n)_2)\, K_{n-1})/K_{n-1}$ under the natural isomorphism between H_{n-1} and F_{n-1}/K_{n-1}. Define

$$C(\theta(n)_1, \theta(n)_2) := A_n$$

$$H_n := \langle H_{n-1}, x_n \mid x_n a x_n^{-1} a^{-1} = 1,\ \forall a \in A_n \rangle.$$

Let A_{n+1} be the subgroup of H_n that corresponds to $(B(\theta(n+1)_1, \theta(n+1)_2)\, K_n)/K_n$ under the natural isomorphism between H_n and F_n/K_n. Define

$$H := \bigcup_{n=1}^{\infty} H_n.$$

Note that this definition of H does not rule out the possibility of A_i, A_j being the same when $i \neq j$ even though $B(\theta(i)_1, \theta(i)_2)$, and $B(\theta(j)_1, \theta(j)_2)$ are different in F_∞. But this will pose no problem to the definitions below nor to our final goals contained in Theorem 1.

Let

$$w_1, w_2, \ldots.$$

be a listing of *all* elements of H such that for each i_0, there are infinitely many j for which $w_j = w_{i_0}$ in H. One can do this for instance by enumerating the set

$$\{s_{ij} \mid s_{ij} \in H,\ s_{ij} = s_{ik},\ \forall i, j, k = 1, 2, 3, \ldots\}.$$

(b) LEMMA 2. *Let $u \in H_m$, where m is a natural number. If there exists a finite permutation representation $(\bar{H}_m, \Gamma_m)$ of H_m where $\bar{u} \neq 1$, then, for every natural number r, such a representation exists for u^r too. i.e., $\hat{u}^r \neq 1$ in some finite permutation representation $(\hat{H}_m, \mathscr{K}_m)$ of H_m.*

Proof. We shall use induction on m.

Since H_1 is free and thus torsion-free and residually finite, it follows that the lemma holds when $m = 1$. Let $n \geqslant 2$ and assume the lemma holds for $1 \leqslant m \leqslant n-1$. We shall prove that this implies that the lemma holds for $m = n$. Let A_+ be the collection of all elements a in H_{n-1} such that $\bar{a} \in \bar{A}_n$ in all finite permutation representations $\bar{H}_{n-1}$ of H_{n-1}. The definitions of A_n are given above in this section.

For every finite permutation representation $\bar{H}_{n-1}$ of H_{n-1}, define the (abstract) group $\hat{H}_n$ as

$$\hat{H}_n := \langle \bar{H}_{n-1}, x_n \mid x_n \bar{a} x_n^{-1} a^{-1} = 1,\ \forall \bar{a} \in \bar{A}_n \rangle. \tag{1}$$

Then $\hat{H}_n$ is a quotient group of H_n, and also contains H_{n-1}. Let E be the collection of all elements h in H_n such that for every finite permutation representation $\bar{H}_{n-1}$ of H_{n-1}, $\hat{h}$ and $\hat{u}$ are conjugate in $\hat{H}_n$. Element u appears in the statement of this lemma. Note that our definitions of A_+ and $\hat{H}_n$ imply that $x_n \bar{b} x_n^{-1} \bar{b}^{-1} = 1$ for all $b \in A_+$. Since $u \in H_n$ and $\hat{H}_n$ is a quotient group of H_n, then E contains the representation of u in $\hat{H}_n$. So, our definition of E shows that $E \neq \varnothing$ and contains some h_0 which has one of the following forms:

$$\begin{aligned} h_0 &= v_0 \\ h_0 &= x_n^{e_1} \\ h_0 &= v_1 x_n^{e_1} v_2 x_n^{e_2} \cdots v_p x_n^{e_p}, \end{aligned} \tag{2}$$

where $v_0 \in H_{n-1}$, $v_1, v_2, ..., v_p \in H_{n-1} \backslash A_+$, $e_1, e_2, ..., e_p$ are non-zero integers and p is a natural number.

Let $\bar{H}_n$ be some finite permutation representation of H_n such that $\bar{u} \neq 1$. Representation $\bar{H}_n$ exists by hypothesis of this lemma. Let $\bar{H}_{n-1}$ be the corresponding finite permutation representation of H_{n-1} in $\bar{H}_n$, and let $\hat{H}_n$ be the corresponding extension defined in (1). Let us note that with this definition, $\bar{H}_n$ becomes a quotient group of $\hat{H}_n$, and as before $\hat{H}_n$ is a quotient group of H_n. So, since $\bar{u} \neq 1$ in $\bar{H}_n$, then $\hat{u} \neq 1$ in $\hat{H}_n$ and so $\hat{h}_0 \neq 1$ in $\hat{H}_n$.

If $h_0 \in H_{n-1}$, then by inductive hypothesis, there exists a finite permutation representation $(H'_{n-1}, \Gamma'_{n-1})$ such that $(h_0^r)' \neq 1$. If we define x'_n as identity map on Γ'_{n-1}, then in the finite permutation representation $\langle H'_{n-1}, x'_n \rangle$ for H_n, we have $(h_0^r)' \neq 1$. From definition of h_0, we then have $(u^r)' \neq 1$ — which is the assertion of the lemma for H_n.

Suppose then that $h_0 \notin H_{n-1}$. In that case, h_0 has form $(2)_2$ or $(2)_3$. Suppose it has form $(2)_3$. Then

$$h_0^r = v_1 x_n^{c_1} v_2 x_n^{c_2} \cdots v_q x_n^{c_q}, \tag{3}$$

where $v_1, v_2, ..., v_q \in H_{n-1} \backslash A_+$, $c_1, c_2, ..., c_q$ are non-zero integers and q is a natural number. The property of the elements v_i implies that for each i, there exists a finite permutation representation $H'_{n-1,i}$ of H_{n-1} acting on some finite set Γ_i for which $v'_i \in H'_{n-1,i} \backslash A'_+$. If (H''_{n-1}, Γ'') is the Cartesian product of the permutation representations $(H'_{n-1,i}, \Gamma_i)$ with Γ'' being the Cartesian product of the sets Γ_i, then $v''_i \in H''_{n-1} \backslash A''_+$ for all i. Put differently, there exists a normal subgroup L_{n-1} of H_{n-1} of finite index such that $v_i \in H_{n-1} \backslash (A_+ L_{n-1})$ for all i.

Let $M := 1 + \sum_{i=1}^{q} |c_i|$. Define the action $*$ of H_n on $H_{n-1}/L_{n-1} \times \{1, 2, ..., M\}$ as follows:

$$\begin{aligned} &(vL_{n-1}, j) * h \\ &\qquad := (vhL_{n-1}, j), \qquad \forall v, h \in H_{n-1} \\ &(v_1 L_{n-1}, j) * x_n^{c_1/|c_1|} \\ &\qquad := (v_1 L_{n-1}, j+1), \qquad \text{for} \quad j = 1, 2, ..., |c_1| \end{aligned} \tag*{$(4)_1$}$$

$$\begin{aligned}
&(v_1 v_2 L_{n-1}, j) * x_n^{c_2/|c_2|}\\
&\quad := (v_1 v_2 L_{n-1}, j+1) \qquad \text{for} \quad j = 1+|c_1|, 2+|c_1|, ..., 1+|c_1|+|c_2|\\
&(v_1 v_2 \cdots v_i L_{n-1}, j) * x_n^{c_i/|c_i|}\\
&\quad = (v_1 v_2 \cdots v_i L_{n-1}, j+1)\\
&\quad \text{for} \quad j = 1+\textstyle\sum_{k=1}^{i-1} |c_k|, 2+\sum_{k=1}^{i-1} |c_k|, ..., 1+\sum_{k=1}^{i} |c_k|,\\
&\quad i = 3, 4, ..., q. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4)_2
\end{aligned}$$

Note that $(v_1 L_{n-1}, j)$, $(v_1 v_2 L_{n-1}, j)$ are in different orbits of action of A_n since $v_2 \notin A_+ L_{n-1} \supseteq A_n L_{n-1}$. Similar arguments show that the action of x_n so far defined in (4) is compatible with the definition of x_n in H given in Section 2(a). In fact, in definition (4), the action of x_n has been defined on at most one point in every orbit of A_n. Since in the representation $(H_{n-1}, H_{n-1}/L_{n-1})$ the orbits of A_n are all isomorphic as far as the action of A_n goes, we then see that the action of x_n can be completed in such a way as to centralise that of A_n. Thus $(1L_{n-1}, 1) * h_0^r \neq (1L_{n-1}, 1)$ and so $\bar{h}_0^r \neq 1$ in the finite permutation representation just given. And from the definition of E at the beginning of this proof, $\bar{h}_0^r$ and $\bar{u}^r$ will be conjugate in the permutation action of H_n just defined. Thus $\bar{u}^r \neq 1$ in this finite permutation representation.

If h_0 has form $(2)_2$, then we merely choose a finite permutation action $\bar{H}_n$, where $\bar{x}_n^{e_1 r} \neq 1$ and $\bar{v} = 1$ for all $v \in H_{n-1}$. Such a choice is compatible with the definition of x_n in H_n contained in Section 2(a). Q.E.D.

COROLLARY 3. *With definition* (1) *above, let* $u_1, u_2, ..., u_k \in H_n$ *for some natural numbers k and n. Suppose there exists a finite permutation representation* (H_n', Γ') *such that* $u_1', u_2', ..., u_k'$ *are non-identity permutations. Then for any natural numbers* $r_1, r_2, ..., r_n$, *there exists a finite regular permutation representation* $(\bar{H}_n, \bar{\Gamma})$ *such that* $\bar{u}_1^{r_1}, \bar{u}_2^{r_2}, ..., \bar{u}_k^{r_k}$ *are all non-identity permutations.*

Proof. Assume the hypotheses of the corollary. By Lemma 2, for every natural number i in $\{1, 2, ..., k\}$, there exists a finite permutation representation (H_n'', Γ_i'') of H_n, where $(u_i^{r_i})'' \neq 1$. The induced action of H_n on the Cartesian product of the sets Γ_i'' gives us a finite permutation representation $(\bar{H}_n, \Gamma)$ of H_n, where $\bar{u}_i^{r_i} \neq 1$ for all i. In that case, the representation $(\bar{H}_n, \bar{H}_n)$, where action is right multiplication gives us a regular finite permutation of H_n, where $\bar{u}_i^{r_i} \neq 1$ for all i. Q.E.D.

3. Idea of Proof of Theorem 1

The idea is similar to the idea of proof of [1, Theorem 4] but with some differences. One difference is that this method uses the stronger result in Lemma 2 above.

We start with a finite permutation group and successively augment the group with the centralisers of its finitely generated subgroups. With the augmented group at each stage appear new finitely generated subgroups whose centralisers are in turn augmented. The permutation groups at each stage are finite. To obtain torsion-free property, the words in the augmented groups are enumerated and, by use of Lemma 2, we ensure that a new word which is a power of an earlier word has an identity permutation action only if that earlier one has such identity action. The rest of the proof is organization of what comes when.

Proof of Theorem 1. We shall use the definitions contained in Section 2(a). Let (y_i) $i \in \mathbb{N} \setminus \{1\}$ be a sequence of symbols. Let us explain some notations which appear below. If g is a permutation of some set Σ and $\Gamma \subseteq \Sigma$ is a union of its orbits, $g_{(\Gamma)}$ shall refer to the restriction of g to Γ. In a similar way, $G_{(\Gamma)}$ is defined as the restriction of G to Γ for a permutation group G, where Γ is a union of some orbits of G.

Step 1. Let $(H_{1(\Sigma_1)}, \Sigma_1)$ be a regular finite permutation representation of H_1 contained in Section 2(a).

Suppose, for some *fixed* $p \in \mathbb{N}$, we have $A_p < H_1$. Let

$$A_{p_1}, A_{p_2}, A_{p_3}, \ldots \tag{5}$$

be *the* subsequence of $A_1, A_2, A_3, \ldots$ whose terms are all equal to A_p. The sequence (A_{p_k}) $k \in \mathbb{N}$ is infinite because of the definition of $\underset{\sim}{\theta}$ which enumerates $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and not $\mathbb{N} \times \mathbb{N}$; and so there exist i, j, where $\theta(i)_3 \neq \theta(j)_3$ while $C(\theta(i)_1, \theta(i)_2) = C(\theta(j)_1, \theta(j)_2)$. Suppose $m_1(p)$ is the number of permutations z in $\mathrm{Sym}(\Sigma_1)$ that centralise the representation $A_{p(\Sigma_1)}$ in $H_{1(\Sigma_1)}$. Of course, $m_1(p)$ is finite since $\mathrm{Sym}(\Sigma_1)$ is finite. Now consider the subsequence (p_k) $k \in \mathbb{N}$ defined in (5) and the corresponding symbols (y_{p_k}) $k \in \mathbb{N}$. We now assign to the first $m_1(p)$ symbols in (y_{p_k}), $k = 2, 3, \ldots$ the permutations z.

Perform the operations in the last paragraph *for every* p satisfying $A_p < H_1$. Note that since $A_p < H_{p-1}$ for $p = 2, 3, \ldots$, then y_2 and infinitely many other y_p have thus been assigned.

Now define

$$x_{2(\Sigma_1)} := y_2.$$

Then $x_{2(\Sigma_1)}$ centralises $A_{1(\Sigma_1)}$ within $\mathrm{Sym}(\Sigma_1)$. We note that $(H_{2(\Sigma_1)}, \Sigma_1)$ has thus been defined.

Step 2. Using Corollary 3, we construct a regular finite permutation representation $(H_{2(\Delta_2)}, \Delta_2)$ of H_2 such that

$$\text{if } w_2 = v^r \text{ for some } v \text{ in } H_2,\ r \in \mathbb{N} \text{ and } v_{(\Sigma_1)} \neq 1, \text{ then } w_{2(\Delta_2)} \neq 1.$$

Let $\Sigma_2 := \Sigma_1 \cup \Delta_2$. For each $A_p < H_1$, let $m_2(p)$ be the number of permutations z in $\mathrm{Sym}(\Sigma_2)$ which centralise $A_{p(\Sigma_2)}$. Assign these permutations z to the first yet-unassigned $m_2(p)$ symbols in $(y_{p_k})\ k \in \mathbb{N}$ with $(y_{p_k})\ k \in \mathbb{N}$ still as defined in Step 1. We realise that this implies that we are assigning some permutations which have already been assigned in an earlier step. However, this repetition does not matter in the end. What matters is that every centraliser is assigned at least once.

For each A_p in H_2 but not contained in H_1 define $(A_{p_k})\ k \in \mathbb{N}$, $(y_{p_k})\ k \in \mathbb{N}$ as in Step 1; i.e., the sequence $p_1, p_2, p_3, \ldots$ is the subsequence of 2, 3, 4, ... for which $A_{p_k} = A_p$ for all k. Let $q_2(p)$ be the number of permutations z in $\mathrm{Sym}(\Sigma_2)$ which centralise $A_{p(\Sigma_2)}$, and assign these to the first $q_2(p)$ symbols in (y_{p_k}). With the assignment in this paragraph and with $A_p < H_{p-1}$ for all p, we have that y_3 has been assigned a permutation.

Let $\mathscr{D}_3$ be the domain, and thus range, of y_3. Define

$$x_{3(\mathscr{D}_3)} := y_3$$
$$x_{3(\Sigma_2 \setminus \mathscr{D}_3)} := 1.$$

The set $\Sigma_2 \setminus \mathscr{D}_3$ appears in this definition to allow for the possibility that y_3 was assigned in Step 1. With the definition, $x_{3(\Sigma_2)}$ centralises $A_{3(\Sigma_2)}$; and moreover a finite permutation representation (H_3, Σ_2) has been constructed.

General Step n. Let $n > 1$. Assume a finite permutation representation $(H_{n(\Sigma_{n-1})}, \Sigma_{n-1})$ has been defined. By Corollary 3, there exists a finite regular permutation representation $(H_{n(\Delta n)}, \Delta_n)$ of H_n satisfying

$$\begin{aligned}&\text{for } k = 2, 3, \ldots, n, \text{ if } w_k = v^{r_k}\\ &\text{for some } v \in H_n,\ r_k \in \mathbb{N} \text{ where } v_{(\Sigma_{n-1})} \neq 1, \text{ then } w_{k(\Delta_n)} \neq 1.\end{aligned} \tag{6}$$

Let $\Sigma_n := \Sigma_{n-1} \cup \Delta_n$. Then follow the procedure in Step 2 besides the first paragraph of that step with H_{n-1}, $m_n(p)$, Σ_n, Δ_n, $q_n(p)$, $\mathscr{D}_{n+1}$, y_{n+1}, H_{n+1} in place of H_1, $m_2(p)$, Σ_2, Δ_2, $q_2(p)$, $\mathscr{D}_3$, y_3, H_3, respectively.

Then define

$$\Sigma := \bigcup_{n=1}^{\infty} \Sigma_n$$
$$g_1 := (x_{1(\Sigma_1)}, x_{1(\Delta_2)}, x_{1(\Delta_3)}, \ldots)$$
$$g_n := (x_{n(\Sigma_{n-1})}, x_{n(\Delta_n)}, x_{n(\Delta_{n+1})}, \ldots) \quad \text{for} \quad n = 2, 3, 4 \cdots$$
$$G := \langle g_i \mid i \in \mathbb{N} \rangle \subseteq \mathrm{Sym}(\Sigma).$$

Conclusion. We conclude the proof of the theorem by showing that (G, Σ) satisfies all the properties claimed by the theorem.

Torsion-Free Property and Sharpness. Let $z \in G \setminus \{1\}$. Then $z = w(g_1, g_2, ..., g_m)$ for some $w(x_1, x_2, ..., x_m) \in H_m$. By definition of maps g_i, we have that Σ_m, Δ_i $i > m$ are all unions of orbits of z. Since $z \neq 1$, then $z_{(\Sigma_p)} \neq 1$ for some $p > m$. Let r be an arbitrary positive integer. Also, since $z \neq 1$, then $w(x_1, x_2, ..., x_m) \neq 1$ in H_m and by Corollary 3, $[w(x_1, x_2, ..., x_m)]^r \neq 1$ in H_m. Since in our listing of (w_i) in Section 2(a), each term occurs infinitely many times then $w_i = [w(x_1, x_2, ..., x_m)]^r$ for some $i > p > m$. Our construction implies immediately that $z^r_{(\Sigma_i)} \neq 1$, and even more so we have the set $\mathrm{fix}(z^r)$ of all elements fixed by z^r to be contained in Σ_i. This is because $(H_{k(\Sigma_j)}, \Sigma_j)$ is a regular representation for $j > k$ and because of procedure (6) above. But each Σ_i is finite. Hence $\mathrm{fix}(z^r)$ is finite, and consequently $\mathrm{fix}(z)$ is also finite. So, we conclude that (G, Σ) is both torsion-free and sharp.

Orbits of Finitely Generated Subgroups of (G, Σ). For each n, *note that each of the sets* Σ_n, Δ_{n+1}, Δ_{n+2}, ... is finite, and is a union of orbits of $\langle g_1, g_2, ..., g_n \rangle$. Since G is generated by the elements g_i, then the orbits of every finitely generated subgroup of (G, Σ) are finite sets.

Relative Homogeneity of (G, Σ). Let L be a finitely generated subgroup of G and let z be an isomorphism on a finite number of orbits of L on Σ. If $z = 1$, then $1_{(\Sigma)}$ is an element which extends z and centralises L. So assume $z \neq 1$. From the last paragraph on orbits of finitely generated subgroups, we see that the domain and range of z are contained in some $\Sigma_n \cup \Delta_{n+1} \cup \cdots \cup \Delta_{n+k}$ or for short in some Σ_j. It is well known that such z will be a restriction of a map h in $\mathrm{Sym}(\Sigma_j)$ which centralises L. By our construction of (G, Σ), we have that $h = y_s$ for some s with $A_s = L$. Hence h, and therefore z, is a restriction of g_s. Therefore z extends to a map g_s in G which centralises L.

Faithfulness of (G, Σ). Elements of G are permutations on Σ and thus (G, Σ) is faithful. Q.E.D.

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Reference

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